Many-valued logics

Marco Degano

Philosophical Logic 2025
29 October 2025

Plan

1. Sorites and i
2. n-valued logics
3. Logic of Paradox
4. Higher-order Vagueness
5. Fuzzy Logics

Readings

Suggested:

- Lecture notes: ch. 3.2.2; ch. 3.3.3 ch. 4.1

Further reading:

- An Introduction to Non-Classical Logic (Priest): ch. 7.4, 7.5; ch. 11
- Logic for Philosophy (Sider): ch. 3.4.4-3.4.5
- Philosophical Logic (MacFarlane): ch. 8.3

Outline

The Sorites revisited

Recall the structure of the Sorites paradox:

Left:

p_N

$p_N \to p_{N-1}$

$\vdots$

$p_2 \to p_1$

$\therefore\ p_1$

Right:

p_N

$p_N \to p_{N-1}$

$\ldots$

$p_{k-1} \to p_k$

$p_k \to p_{k-1}$

$\ldots$

$p_l \to p_{l-1}$

$\ldots$

$p_2 \to p_1$

$\therefore\ p_1$

- Using classical logic (left), the conclusion must be true.
- Using K_3^s (right), we can make some of the premises neither true nor false, and the conclusion false.
- What would be the situation with Ł3?

The K_3^s answer

The K_3^s answer to the sorites paradox is: **we reject some of the premises as not true.**

Philosophical discussion

- **Tolerance vs. local failure:** Three-valued logics block Sorites by letting some tolerance links be *indeterminate*. Yet ordinary intuition treats each link as (seems) true. Revise logic, or revise the intuition?
- **Arbitrariness:** The boundary between 1 and i and i and 0 feels arbitrary and not precise.
- **Assertion:** If indeterminate statements are neither true nor false, what licenses asserting them or relying on them in decisions?
- **Semantic vs Ontic**: What is indeterminate: language or world? If the world is perfectly precise (epistemicism), truth-value gaps look misguided. If the world is metaphysically vague, that's controversial (your 2nd assignment).

Outline

n-valued logics: why go beyond three?

- We may replace the single value i (or $\frac{1}{2}$) with multiple intermediate degrees to achieve finer granularity.
- Sorites links evaluate *close to* 1 (e.g. $0.99, 0.98, \ldots$) rather than landing at $\frac{1}{2}$.
- At each step, the truth degree drops slightly, matching the "small change, small effect" intuition.
- Our task is to extend a 3-valued matrix to an n-valued semantics.

Recalling definitions

- Given a language $\mathcal{L}$, the general make-up of a finite many-valued logic will be formed by
  1. A finite non-empty set of truth values T
  2. A set $T^+ \subseteq T$ of designated truth values
  3. For each n-place connective, a truth value function $v : T^n \to T$. If $n = 0$, $v(\cdot) \in T$
- These three elements form the **logical matrix** of $\mathcal{L}$.
- Given a set of formulas Γ and a formula ϕ, we say that Γ **entails** ϕ (written $\Gamma \models \phi$) iff for every valuation v, whenever $v(\gamma) \in T^+$ for all $\gamma \in \Gamma$, it follows that $v(\phi) \in T^+$

n-valued logics

- For $n \geq 2$, an n-valued logic is defined over the set of truth-values:

$$T_n = \left\{ \frac{k}{n-1} : k = 0, 1, \ldots, n-1 \right\} \subseteq [0, 1]$$

- We take $T^+ = \{1\}$.
- Different truth-value functions for the connectives give rise to different logics.
- In particular, we can extend K_3^s and Ł3 to K_n^s and Łn by extending pointwise the same semantic clauses (e.g., conjunction defined as the minimum of the values).

Sorites in n-valued Strong Kleene

- 100 items $m = 1, \ldots, 100$ with step $\delta = \frac{1}{99}$:

$$v(p_1) = 1, \qquad v\big(p_{m+1}\big) = v\big(p_m\big) - \delta \;\; \Rightarrow \;\; v(p_m) = 1 - (m-1)\delta$$

- **Strong Kleene** connectives

$$\neg a = 1{-}a, \quad a{\wedge}b = \min(a,b), \quad a{\vee}b = \max(a,b), \quad a \to b = \max(1{-}a,\, b)$$

- For $m = 1, \ldots, 99$

$$\begin{aligned} v\big(p_m \to p_{m+1}\big) &= \max\big(1 - v(p_m),\, v(p_{m+1})\big) \\ &= \max\Big(\tfrac{m-1}{99},\, 1 - \tfrac{m}{99}\Big) \end{aligned}$$

- Hence $v(p_1 \to p_2) = \frac{98}{99}$, the values dip to a minimum $\frac{49}{99}$ at $m = 50$, then rise back to $\frac{98}{99}$ at $m = 99$.
- All links are < 1 (the Sorites series is blocked)

Sorites with Łukasiewicz implication

- Same profile: $v(p_m) = 1 - \frac{m-1}{99}$
- **Łukasiewicz implication:** $a \Rightarrow b = \min\big(1,\ 1 - a + b\big)$.
- For $m = 1, \ldots, 99$,

$$\begin{aligned} v\big(p_m \Rightarrow p_{m+1}\big) &= \min(1,\ 1 - v(p_m) + v(p_{m+1})) \\ &= \min\big(1,\ 1 - \big[1 - \tfrac{m-1}{99}\big] + \big[1 - \tfrac{m}{99}\big]\big) = 1 - \tfrac{1}{99} = \tfrac{98}{99} \end{aligned}$$

- So *every* tolerance step is very nearly true, but strictly below 1.
- With n items (step $\delta = \frac{1}{n-1}$), each Łukasiewicz link has value $1 - \delta$ (still < 1). In Strong Kleene, link values range from $1 - \delta$ down to $\frac{1}{2} - \frac{\delta}{2}$ and back.
- In both cases, the argument is blocked as the premises are not fully true.
- Which one is more faithful to our intuitions?

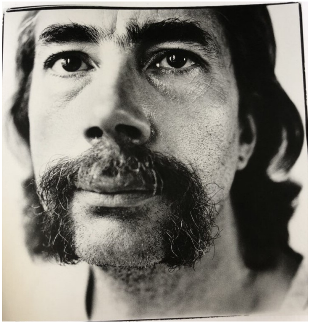
Philosophical discussion

- How many values n? Equal spacing or fitted curves? What fixes the step δ and any threshold(s)? The sharp line is replaced by parameters that still need justification.
- Do we really want *degrees of truth*? Or are we modeling degrees of belief/evidence/assertability instead? If truth is graded, what is the scale type (ordinal/interval/ratio)?
- Are degrees commensurable across predicates? e.g. is *Amsterdam is beautiful* "truer" than *New York is big*? If comparability fails, a single numerical scale for all sentences may mislead.

[Williamson 2002]

Outline

The logic of paradox (LP)



Graham Priest

- We now consider a different answer to the Sorites: the premises are true, but *Modus Ponens* is not a valid rule of inference.
- We have always assumed that $T^+ = \{1\}$.
- The Logic of Paradox (LP) takes $T^+ = \{1, i\}$ with Strong Kleene semantics.

[Priest 2008]

Paraconsistent vs Paracomplete

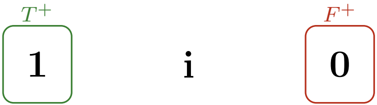
- A logic with consequence relation $\models$ is *paraconsistent* iff there exist formulas ϕ, ψ such that $\{\phi, \neg\phi\} \not\models \psi$.
- A logic with $\models$ is *paracomplete* iff there exists a formula ϕ such that $\not\models \phi \vee \neg\phi$.
- LP (with $T^+ = \{1, i\}$ and Strong Kleene tables) is paraconsistent: $\{p, \neg p\} \not\models_{\mathsf{LP}} q$.
- K_3^s (with $T^+ = \{1\}$) is paracomplete: $\not\models_{K_3^s} p \vee \neg p$

LP: gaps and gluts

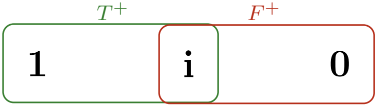
- Given a set of truth-values T, we take $T^+ \subseteq T$ for the truth and $F^+ \subseteq T$ for the falsity.
- ***Gap:*** a value t is a gap iff $t \notin T^+$ and $t \notin F^+$.
- ***Glut:*** a value t is a glut iff $t \in T^+$ and $t \in F^+$

$$T^+_{\mathrm{K3}} = \{1\}, \quad F^+_{\mathrm{K3}} = \{0\}; \qquad T^+_{\mathrm{LP}} = \{1, i\}, \quad F^+_{\mathrm{LP}} = \{0, i\}$$

- In K3, i is a *gap* (neither true nor false).



- In LP, i is a *glut* (both true and false).



Some facts

- Modus ponens fails:

$$p, p \to q \not\models_{\mathsf{LP}} q$$

- The set of tautologies in LP coincides with the set of classical tautologies.

Theorem (LP-Classical Validity Equivalence)

Let LP be the propositional logic using the Strong Kleene truth-functions on $\{0, i, 1\}$ with designated values $T^+_{\mathsf{LP}} = \{1, i\}$. Then for every propositional formula ϕ,

$$\models_{\mathsf{LP}} \phi \text{ iff } \models_{\mathsf{CL}} \phi$$

Two lemmas

Lemma 1 (Refinement)

Fix the Strong Kleene truth-functions on $\{0, i, 1\}$, and let $v, w : P \to \{0, i, 1\}$ be two valuations. Define $v \leq w$ (w *refines* v) iff for every atom p: $v(p) = 0 \Rightarrow w(p) = 0$ and $v(p) = 1 \Rightarrow w(p) = 1$. Then for every formula ϕ,

$$v(\phi) \in \{0, 1\} \implies w(\phi) = v(\phi)$$

Lemma 2 (Classicality)

Let $v : P \to \{0, 1\}$ be a *classical* valuation. Write v_{SK} for the extension using the Strong Kleene truth-functions and v_{CL} for the classical (two-valued) extension. Then, for every formula ϕ,

$$v_{\mathrm{SK}}(\phi) = v_{\mathrm{CL}}(\phi)$$

Proof of Theorem

(⇒) Assume $\not\models_{\mathrm{CL}} \phi$. Then there exists a classical valuation v with $v_{\mathrm{CL}}(\phi) = 0$. By Lemma 2, $v_{\mathrm{SK}}(\phi) = 0$. Hence $v_{\mathrm{LP}}(\phi) = 0$ (same truth-functions), and since $0 \notin T^{+}_{\mathsf{LP}}$, ϕ is not LP-valid. Thus, $\models_{\mathrm{LP}} \phi \Rightarrow \models_{\mathrm{CL}} \phi$.

(⇐) Assume $\not\models_{\mathrm{LP}} \phi$. Then there exists a (SK/LP) valuation v with $v(\phi) = 0$. Define a *classical refinement* $w \geq v$ by sending each atom p with $v(p) = i$ to either 0 or 1, and keeping $0, 1$ fixed. Since $v(\phi) = 0 \in \{0, 1\}$, Lemma 1 gives $w(\phi) = 0$. By Lemma 2, $w_{\mathrm{CL}}(\phi) = 0$, so $\not\models_{\mathrm{CL}} \phi$. Thus, $\models_{\mathrm{CL}} \phi \Rightarrow \models_{\mathrm{LP}} \phi$.

Therefore $\models_{\mathsf{LP}} \phi$ iff $\models_{\mathsf{CL}} \phi$. □

Assessing the Situation

The $Ł_n/K^s_n$ answer to the sorites paradox is: **we reject some of the premises as not true.**

The LP answer to the sorites paradox is: **the argument is not valid (modus ponens fails).**

Philosophical discussion

- LP's semantics tolerates *gluts* (i is both true and false). One may read this metaphysically (dialetheism) or instrumentally (safe reasoning with inconsistent information).
- LP validates all classical tautologies, yet blocks *Explosion* ($p, \neg p \not\models q$).
- Relative to K3 (gappy), LP keeps Excluded Middle and tolerates gluts. Which "departure" from classical logic (gaps vs. gluts) is the better cost for the target phenomena?

Outline

Higher-order vagueness: the core worry

- Three-valued accounts soften the sharp $1/0$ cut by adding a indeterminate value i.
- But then two interfaces remain: $1/i$ and $i/0$.
- When we talk about what is *definitely* true/false, do these interfaces themselves become sharp again?

$$\mathbf{1} \quad \Big| \quad \mathbf{i} \quad \Big| \quad \mathbf{0}$$

interface   interface

- Even with i, adding *definitely* risks re-introducing *sharp* and *arbitrary* boundaries.

Determinacy as *crisp*

We interpret "definitely" as a determinacy operator Δ. Let v be a K_3^s valuation on $\{0, i, 1\}$. We take:

$$v(\Delta\phi) = \begin{cases} 1 & \text{iff } v(\phi) = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad \nabla\phi := \neg\Delta\phi \wedge \neg\Delta\neg\phi$$

- $\Delta\phi$ says: ϕ is *definitely* true.
- Δ is a *filter* for full truth.
- $\nabla\phi$ says: ϕ is not *definitely* true and not *definitely* false (first-order vagueness).

p	Δp	∇p
1	1	0
i	0	1
0	0	0

Basic laws for Δ (with SK connectives)

- **T (factiveness):** $\Delta\phi \to \phi$ is valid.
- **K (distribution over $\to$):** $\Delta(\phi \to \psi) \to (\Delta\phi \to \Delta\psi)$ is valid.
- **4 (positive introspection):** $\Delta\phi \to \Delta\Delta\phi$ is valid.
- **Collapse:** $\Delta\Delta\phi \leftrightarrow \Delta\phi$ (no growth of determinacy past one application).

The problem of higher-order vagueness

Write $\Delta^1\varphi := \Delta\varphi$ and $\Delta^m\varphi := \Delta(\Delta^{m-1}\varphi)$ for $m \geq 2$.

Define first-order vagueness:

$$\nabla^{(1)}\varphi \ := \ \neg\Delta\varphi \ \wedge \ \neg\Delta\neg\varphi$$

Define mth-order vagueness (so that *second order* really is $\nabla(\Delta\varphi)$):

$$\nabla^{(m)}\varphi \ := \ \nabla(\Delta^{m-1}\varphi) \quad (m \geq 2)$$

- **Stability:** $\Delta^{m+1}\varphi \leftrightarrow \Delta^m\varphi$ for all $m \geq 1$ (collapse of Δ).
- **No second-order vagueness:** $\nabla^{(2)}\varphi \ = \ \nabla(\Delta\varphi) \ = \ 0$ (hence $\nabla^{(m)}\varphi = 0$ for all $m \geq 1$).
- First-order vagueness may remain ($\nabla^{(1)}\varphi$ can be 1), but once something is *definitely* true/false, there are no borderline cases of that.

Outline

Truth and Degrees*

- We introduced n-valued logics using a finite set of truth values. Truth-value domains need *not* be finite:
  - Finite-valued:
  $$T_n = \{0, \tfrac{1}{n-1}, \tfrac{2}{n-1}, \ldots, 1\},\ n \in \mathbb{N}, n \geq 2$$
  - Rational-valued:
  $$T_{\aleph_0} = \{\tfrac{m}{n} : 0 \leq m \leq n,\ m, n \in \mathbb{N},\ n \neq 0\}$$
  - Real-valued:
  $$T_{\aleph_1} = [0, 1]$$
- To get a many-valued logic, we choose:
  - a truth-value set (finite or continuous);
  - how connectives act on degrees (e.g. $\wedge = \min$, $\vee = \max$, $\neg x = 1 - x$, and a suitable $\rightarrow$)
  - truth-preserving vs. degree-preserving logical consequence.

*We are making an important assumption in the notation used in this slide. Which one? ;-)

Fuzzy Logics and Logical Consequence

Given a set of truth values T with $1 \in T$:

Truth-preserving consequence $\models_1$

$$\Gamma \models_1 \psi \quad \text{iff} \quad \forall v \left(\forall \gamma \in \Gamma,\ v(\gamma) = 1\right) \Rightarrow v(\psi) = 1.$$

Intuition: preserves absolute truth (1).

Degree-preserving consequence $\models_{\mathrm{deg}}$

$$\Gamma \models_{\mathrm{deg}} \psi \iff \forall t \in T \left(\forall v\big[(\forall \gamma \in \Gamma,\ v(\gamma) \geq t) \ \Rightarrow\ v(\psi) \geq t\big]\right)$$

Intuition: reasoning must hold *uniformly at every level of certainty*.

A simple case

Let $T = \{0, \frac{1}{2}, 1\}$. Use Strong Kleene semantics.

Truth-preserving ($\models_1$):

$$\Gamma \models_1 \psi \quad \Longleftrightarrow \quad \Gamma \models_{K_3^s} \psi$$

Degree-preserving ($\models_{\mathrm{deg}}$):

$$\Gamma \models_{\mathrm{deg}} \psi \quad \Longleftrightarrow \quad \Gamma \models_{\mathsf{LP}} \psi \text{ and } \Gamma \models_{K_3^s} \psi$$

Thresholds $t \in \{0, \frac{1}{2}, 1\}$; $t = 0$ vacuous, $t = \frac{1}{2}$ matches LP's $T^+ = \{\frac{1}{2}, 1\}$, and $t = 1$ matches K_3^s's $T^+ = \{1\}$

Modus Ponens

Modus Ponens under $\models_{\text{deg}}$ over $T = [0, 1]$ fails

- *Strong Kleene $\rightarrow$:* take $t = 0.5$, $v(A) = 0.5$, $v(B) = 0.25$. Then $v(A \rightarrow B) = \max(1 - 0.5,\ 0.25) = 0.5 \geq t$ but $v(B) = 0.25 < t$.
- *Łukasiewicz $\Rightarrow$:* take $t = 0.5$, $v(A) = 0.5$, $v(B) = 0.25$. Then $v(A \Rightarrow B) = \min(1,\ 1 - 0.5 + 0.25) = 0.75 \geq t$ but $v(B) = 0.25 < t$.

Degree-preserving entailment as an inf-bound over $T = [0, 1]$

Fact (assume $\Gamma \neq \varnothing$)

$$\Gamma \models_{\mathrm{deg}} \varphi \quad \Longleftrightarrow \quad \forall v \left(\inf\{\, v(\psi) \mid \psi \in \Gamma \,\} \;\leq\; v(\varphi) \right)$$

Recall (degree-preserving): $\Gamma \models_{\mathrm{deg}} \varphi$ iff for all valuations v and all thresholds $t \in [0, 1]$:

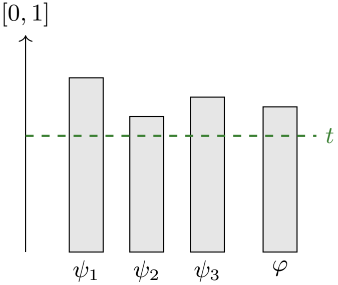
$$\left(\forall \gamma \in \Gamma \;\; v(\gamma) \geq t\right) \implies v(\varphi) \geq t$$

Remark. If you adopt $\inf \varnothing = 1$, the equivalence also holds for $\Gamma = \varnothing$ and yields $\varnothing \models_{\mathrm{deg}} \varphi$ iff $v(\varphi) = 1$ for all v.

Graphical intuition: thresholds vs. the infimum bar

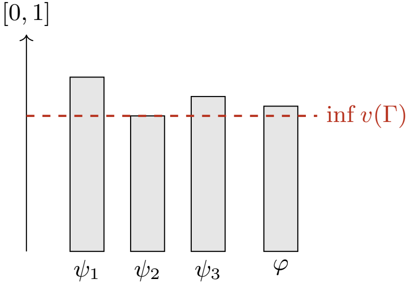
Threshold view:

- Draw any horizontal threshold t.
- If all premise bars are at/above t, then the conclusion bar must also be at/above t.



Infimum view:

- Let the red dashed line be $s = \inf v(\Gamma)$.
- Then $v(\varphi)$ must be at least s.



Fuzzy Logics and the Sorites

- Fix a finite chain $N \geq 2$ and choose any monotone profile $f : \{1, \ldots, N\} \to [0,1]$ with $f(1) = 1 > \cdots > f(N) = 0$. Set $v(p_m) := f(m)$.
- **Truth-preserving** ($\models_1$): under Strong Kleene and Łukasiewicz semantics, each tolerance link is < 1.

> The $\models_1$ answer to the sorites paradox is: **we reject some of the premises as not true**.

- **Degree-preserving** ($\models_{\mathrm{deg}}$): under Strong Kleene and Łukasiewicz semantics, MP fails.

> The $\models_{\mathrm{deg}}$ answer to the sorites paradox is: **the argument is not valid (modus ponens fails)**.

Philosophical discussion

- Again, what is graded truth?
- Truth vs. probability. Degrees of truth are not credences, as they compose with connectives differently. If you want beliefs, add a separate probabilistic layer.
- Rate of the decline: linear vs. logistic vs. stepwise drops. Context effects and subject variability. Is it empirically testable in psycholinguistics experiments?
- Higher-order vagueness: degrees avoid a sharp $1/0$ cut, but a crisp "definitely" raises the same issues as the finite approach.

Addendum: On the relevance of $T_{\aleph_1}$

- To complete the answer to a question which was asked at the very end of the lecture.
- First, if we allow $T_{\aleph_1}$ (all real truth degrees in $[0, 1]$), we can genuinely speak of $v(\phi) = \sqrt{2}$ and allow non-linear truth operations such as $\Delta(x) = \sqrt{x}$ that are not confined to rational values.
- Second, $T_{\aleph_1}$-semantics and $T_{\aleph_0}$-semantics can induce different consequence relations. For instance, we can work in standard Łukasiewicz semantics and take logical consequence to be preservation of the designated value 1.
- Fact: If Γ is *finite*, then

$$\Gamma \models_{\aleph_0} \varphi \quad \text{iff} \quad \Gamma \models_{\aleph_1} \varphi.$$

- If Γ is *infinite*, the left-to-right direction

$$\Gamma \models_{\aleph_0} \varphi \;\Rightarrow\; \Gamma \models_{\aleph_1} \varphi$$

  can fail.

Addendum: On the relevance of $T_{\aleph_1}$

- Idea of the failure: in Łukasiewicz logic you can build formulas that 'pin down' the truth degree of a propositional variable p to lie inside any rational closed interval $[a, b] \subseteq [0, 1]$, with $a, b \in \mathbb{Q}$ (this relates to the optional exercise in the assignment, so I'm not spelling out the construction here. Besides, it is not trivial).
- Now take a countable family of such formulas whose associated intervals $[a_n, b_n]$ are nested and shrink around some fixed irrational $\alpha \in (0, 1)$ (so $\bigcap_n [a_n, b_n] = \{\alpha\}$). Together, these premises force p to take exactly the value α.
  - In $T_{\aleph_1}$ (all reals allowed), we can assign $v(p) = \alpha$, so Γ is satisfiable.
  - In $T_{\aleph_0}$ (only rationals allowed), we *cannot* assign $v(p) = \alpha$ (since α is irrational), so Γ has no satisfying valuation.
- Hence, in $T_{\aleph_0}$ the unsatisfiable Γ vacuously entails every formula φ, while in $T_{\aleph_1}$ the same Γ does *not* entail every φ (the model with $p = \alpha$ can refute some φ). Therefore $\models_{\aleph_0} \neq \models_{\aleph_1}$.
- (Note also that we are taking the language to be (just) countable, which is sufficient to show the difference)